

# On the multivariate Burgers equation and the incompressible Navier-Stokes equation (part III)

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## Abstract

In this paper we first obtain local contraction results in the  $H^m \times H^{2m}$ -norm with  $m \geq 1$  with respect to time and space for a local scheme. Especially, we show that the local scheme preserves  $C^m \cap H^m$  regularity together with polynomial decay of order  $m$  of the data at each time step for each  $m \geq 2$ . For an extended controlled scheme with local limit functions  $l \rightarrow v_i^{\rho,l}(l, \cdot) = v_i^{\rho,l}(l, \cdot) + r_i^l$  we observe linear growth with respect to the time step number  $l$  and with respect to the  $H^m \times H^{2m}$ -norm. We simplify the controlled scheme considered in [1] and [2]. For the simplified controlled scheme we observe linear growth of the squared local limit functions  $l \rightarrow \left(v_i^{\rho,l}(l, \cdot)\right)^2$  measured in  $H^m \times H^{2m}$  norm. This leads to a global linear bound of the Leray projection term. Furthermore, the schemes discussed in [1] and [2] are simplified in the sense that the estimates are achieved without the use of some properties concerning the adjoint of a local fundamental solutions with variable drift terms. We note that the pointwise and absolute convergence of the local functional series and their first order time derivatives and their spatial derivatives leads to a constructive approach of local classical solutions.

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## 1 Introduction

First, we recall a naive simple scheme in brief which leads to local solutions. For time step  $l \geq 1$  and small  $0 < \rho_l \sim \frac{1}{l}$  we consider a time-local functional scheme for functions  $v_i^{\rho,l,k}$ ,  $1 \leq i \leq n, k \geq 0$  with limit

$$v_i^{\rho,l} := v_i^{\rho,l-1}(l-1, \cdot) + \sum_{k=1}^{\infty} \delta v_i^{\rho,l,k}, \quad (1)$$

where for  $1 \leq i \leq n$  we have  $v_i^{\rho,l-1}(l-1, \cdot) \in H^2 \cap C^2$ , and  $\delta v_i^{\rho,l,k} = v_i^{\rho,l,k} - v_i^{\rho,l,k-1}$  are functions defined on the domain  $[l-1, l] \times \mathbb{R}^n$  along with

$v^{\rho,l,-1} := v_i^{\rho,l-1}$ , and where the functions  $v_i^{\rho,l,k}$  satisfy the equations

$$\begin{cases} \frac{\partial v_i^{\rho,l,k}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{\rho,l,k}}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{\rho,l,k-1} \frac{\partial v_i^{\rho,l,k}}{\partial x_j} = \\ \rho_l \sum_{j,m=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{j,m=1}^n \left( \frac{\partial v_m^{\rho,l,k-1}}{\partial x_j} \frac{\partial v_j^{\rho,l,k-1}}{\partial x_m} \right) (\tau, y) dy, \\ \mathbf{v}^{\rho,l,k}(l-1, \cdot) = \mathbf{v}^{\rho,l-1}(l-1, \cdot) \end{cases} \quad (2)$$

on the domain  $[l-1, l] \times \mathbb{R}^n$  for  $l \geq 1$ . Here,  $K_n$  is the fundamental solution of the Laplacian of dimension  $n$ . We are interested in the case  $n \geq 3$ . For  $l = 1$  we define  $\mathbf{v}^{\rho,l-1}(l-1, \cdot) = \mathbf{h}(\cdot)$ . In this part III of articles on the multivariate Burgers equation and the incompressible Navier Stokes equation we make four contributions. First we define a simplified scheme which will allow us to simplify the proof of local contraction in  $H^2$ -based spaces in the sense that the argument based on the adjoint of the fundamental solution is avoided. Second, we improve the result concerning the local contraction by showing that a certain kind of polynomial decay is preserved together with higher order regularity. Third, we explicitly show why the scheme proposed is an approach of constructing classical solution rather than a mere numerical scheme. And fourth, we show that an extension of the scheme with a simplified control function leads  $H^m \times H^{2m}$ -based linear bound of the growth of the solution with respect to the time step number  $l \geq 1$  holds also for the squared norm of the controlled solution function, such that the controlled scheme is really global. The price to pay for a simplified control function is that we have only a linear bound and a linearly decreasing time-step size. We mentioned earlier in [1] that a controlled scheme of the incompressible Navier Stokes equation allows for a constant step size and for a numerical stabilization of computations. Here we propose an alternative proof method for the sake of global existence and regularity. However, we shall see that the essential step is the local contraction estimate.

## 2 A simplified scheme

For  $m \geq 2$  and time step  $l \geq 1$  and small  $0 < \rho_l \sim \frac{1}{l}$  we consider a time-local functional scheme

$$v_i^{*,\rho,l} := v_i^{*,\rho,l-1} + \sum_{k=1}^{\infty} \delta v_i^{*,\rho,l,k}, \quad (3)$$

where for  $1 \leq i \leq n$   $v_i^{*,\rho,l-1}(l-1, \cdot) \in H^m \cap C^m$ , and  $\delta v_i^{*,\rho,l,k} = v_i^{\rho,l,k} - v_i^{*,\rho,l,k-1}$  along with  $v^{*,\rho,l,-1} := v_i^{*,\rho,l-1}$ , and where the functions  $v_i^{*,\rho,l,k}$  sat-

isfy the equations

$$\begin{cases} \frac{\partial v_i^{*,\rho,l,k}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{*,\rho,l,k}}{\partial x_j^2} = -\rho_l \sum_{j=1}^n v_j^{*,\rho,l,k-1} \frac{\partial v_i^{*,\rho,l,k-1}}{\partial x_j} \\ + \rho_l \sum_{j,m=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{j,m=1}^n \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} \frac{\partial v_j^{*,\rho,l,k-1}}{\partial x_m} \right) (\tau, y) dy, \\ \mathbf{v}^{*,\rho,l,k}(l-1, \cdot) = \mathbf{v}^{*,\rho,l-1}(l-1, \cdot). \end{cases} \quad (4)$$

We are interested in the case  $n \geq 3$ , and for  $l = 1$  we define  $\mathbf{v}^{*,\rho,l-1}(l-1, \cdot) = \mathbf{h}(\cdot)$ . Especially, we are interested in the case  $n = 3$  of course. If  $S$  demotes the right side source term of the first equation of (4), then we see that we have the representation

$$\begin{aligned} v_i^{*,\rho,l,k}(\tau, x) &= \int_{\mathbb{R}^3} v^{*,\rho,l-1}(l-1, y) G_l(t, x-y) dy \\ &+ \int_{l-1}^{\tau} \int_{\mathbb{R}^n} S(s, y) G_l(t-s, x-y) dy ds \end{aligned} \quad (5)$$

where  $G_l$  is the fundamental solution of the heat equation  $\frac{\partial u}{\partial t} - \rho_l \nu \Delta u = 0$  on  $[l-1, l] \times \mathbb{R}^n$ . Compared to the previous scheme we observe that the latter expression is a convolution. A priori estimates are easier at hand, and we do not need the adjoint of the fundamental solution in order to shift derivatives. Nevertheless, keeping book of the additional source term we may adopt arguments of part I and part II and arrive at the same result by simpler considerations.

### 3 Statement of local contraction result (strong form)

The functions  $v_i^{\rho,k,l}$  are define locally on  $[l-1, l] \times \mathbb{R}^n$ . For each  $l \geq 1$  we may consider the trivial extension  $v_i^{\rho,k,l} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$  (denoted by the same symbol  $v_i^{\rho,k,l}$  of notation for the sake of simplicity) which is defined to be zero on the complementary domain  $(\mathbb{R} \setminus [l-1, l]) \times \mathbb{R}^n \times \mathbb{R}$ . We may measure these extensions in  $L^2(\mathbb{R}^{n+1}) = L^2(\mathbb{R}) \times L^2(\mathbb{R}^n)$  obviously, where the latter set product notation indicates that time and space may be measured naturally up to different orders of regularity. We have the following local contraction result.

**Theorem 3.1.** *If  $v_i^{\rho,l-1}(l-1, \cdot) \in H^2$  for  $1 \leq i \leq n$ , then for small  $\rho_l > 0$  with  $\rho_l \sim \frac{1}{l}$  in the scheme above we have*

$$\begin{aligned} &\max_{i \in \{1, \dots, n\}} |\delta v_i^{\rho,l,k}|_{H^1 \times H^2} \\ &\leq \frac{1}{2} \max_{i \in \{1, \dots, n\}} |\delta v_i^{\rho,l,k-1}|_{H^1 \times H^2}. \end{aligned} \quad (6)$$

Furthermore we may choose  $\rho_l$  above such that in addition we have

$$\begin{aligned} & \max_{i \in \{1, \dots, n\}} |\delta v_i^{\rho, l, 1}(\tau, \cdot)|_{H^1 \times H^2} \\ &= \max_{i \in \{1, \dots, n\}} |v_i^{\rho, l, 0} - v_i^{\rho, l-1}|_{H^1 \times H^m} \leq \frac{1}{4\sqrt{l}}. \end{aligned} \quad (7)$$

We shall see that this is indeed essential together with linear growth in order to get a global scheme, since for the coefficient function evaluated at  $t \geq 0$  the embedding  $C^\alpha \subset H^2$  for dimension  $n = 3$  (uniformly with respect to time  $t \geq 0$ ) ensures that given the solution this same solution can be represented in terms of fundamental solution of certain scalar parabolic equations which involve the solution in the first order terms. However, in this paper we shall see that we can even improve this as follows. We say that a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $f \in C^m$  (i.e.,  $f$  has continuous partial derivatives up to order  $m$ ) has polynomial decay of order  $m \geq 1$  up to the derivatives of order  $m \geq 1$  if for all multivariate partial derivatives  $D_x^\alpha f$  with  $|\alpha| = \sum_{i=1}^n |\alpha_i|$  and  $|\alpha| \leq m$  we have for all  $x \in \mathbb{R}^n$

$$|D_x^\alpha f(x)| \leq \frac{C_\alpha}{1 + |x|^m} \quad (8)$$

for some  $C_\alpha < \infty$ . Now we can formulate our improvement of theorem 3.1.

**Theorem 3.2.** *If for  $1 \leq i \leq n$  the function  $v_i^{*, \rho, l-1}(l-1, \cdot) \in C^m$  has polynomial decay of order  $m \geq 1$  up to the derivatives of order  $m \geq 1$ , then for all  $k \geq 0$  we have that  $v_i^{*, \rho, k, l}(\tau, \cdot)$  for  $\tau \in [l-1, l]$  is in  $C^m$  and has polynomial decay of order  $m \geq 1$  up to the derivatives of order  $m \geq 1$ . Furthermore for small  $\frac{1}{l} \lesssim \rho_l > 0$  in the scheme we have*

$$\begin{aligned} & \max_{i \in \{1, \dots, n\}} |\delta v_i^{*, \rho, l, k}|_{H^m \times H^{2m}} \\ & \leq \frac{1}{4} \max_{i \in \{1, \dots, n\}} |\delta v_i^{*, \rho, l, k-1}|_{H^m \times H^{2m}}, \end{aligned} \quad (9)$$

and we may choose  $\rho_l$  above such that in addition we have

$$\begin{aligned} & \max_{i \in \{1, \dots, n\}} |\delta v_i^{*, \rho, l, 1}|_{H^m \times H^{2m}} \\ &= \sup_{\tau \in [l-1, l]} \max_{i \in \{1, \dots, n\}} |v_i^{*, \rho, l, 0}(\tau, \cdot) - v_i^{*, \rho, l-1}(l-1, \cdot)|_{H^m \times H^{2m}} \leq \frac{1}{4\sqrt{l}}. \end{aligned} \quad (10)$$

## 4 Proof of theorem 3.2

In order to prove the contraction property (6) we first consider representations of  $\delta v_i^{\rho, l, k} = v_i^{\rho, l, k} - v_i^{\rho, l, k-1}$  in terms of the fundamental solution  $G_l$

defined above. First observe that the equation for  $\delta v_i^{\rho,l,k}$  is

$$\left\{ \begin{array}{l} \frac{\partial \delta v_i^{*,\rho,l,k}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 \delta v_i^{*,\rho,l,k}}{\partial x_j^2} = \\ -\rho_l \sum_{j=1}^n v_j^{*,\rho,l,k-1} \frac{\partial \delta v_i^{*,\rho,l,k-1}}{\partial x_j} - \rho_l \sum_{j=1}^n \delta v_j^{*,\rho,l,k-1} \frac{\partial v_i^{*,\rho,l,k-1}}{\partial x_j} \\ +\rho_l \sum_{j,m=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{j,m=1}^n \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} \frac{\partial v_j^{*,\rho,l,k-1}}{\partial x_m} \right) (\tau, y) dy \\ -\rho_l \sum_{j,m=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{j,m=1}^n \left( \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \frac{\partial v_j^{*,\rho,l,k-2}}{\partial x_m} \right) (\tau, y) dy, \\ \delta \mathbf{v}^{*,\rho,l,k}(l-1, \cdot) = 0. \end{array} \right. \quad (11)$$

Again this equation is considered on the domain  $[l-1, l] \times \mathbb{R}^n$ . Next in terms of the fundamental solution  $G_l$ , and for all  $1 \leq i \leq n$  we have the representation

$$\begin{aligned} \delta v_i^{*,\rho,l,k}(\tau, x) = & \\ & +\rho_l \int_{l-1}^{\tau} \int_{\mathbb{R}^n} \left( -\sum_{j=1}^n \delta v_j^{*,\rho,l,k-1} \frac{\partial v_i^{*,\rho,l,k-1}}{\partial x_j} \right) (s, y) G_l(\tau-s; x-y) dy ds \\ & +\rho_l \int_{l-1}^{\tau} \int_{\mathbb{R}^n} \left( -\sum_{j=1}^n v_j^{*,\rho,l,k-1} \frac{\partial \delta v_i^{*,\rho,l,k-1}}{\partial x_j} \right) (s, y) G_l(\tau-s; x-y) dy ds \\ & +\rho_l \int_{l-1}^{\tau} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(z-y) \right) \times \\ & \times \left( \sum_{m,j=1}^n \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} + \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \right) \right) (s, y) G_l(\tau-s, x-z) dy dz ds, \end{aligned} \quad (12)$$

where for convenience we rewrite the difference of the Leray projection terms observing that

$$\begin{aligned}
& \sum_{j,m=1}^n \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} \frac{\partial v_j^{*,\rho,l,k-1}}{\partial x_m} \right) - \sum_{j,m=1}^n \left( \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \frac{\partial v_j^{*,\rho,l,k-1}}{\partial x_m} \right) \\
& + \sum_{j,m=1}^n \left( \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \frac{\partial v_j^{*,\rho,l,k-1}}{\partial x_m} \right) - \sum_{j,m=1}^n \left( \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \frac{\partial v_j^{*,\rho,l,k-2}}{\partial x_m} \right) \\
& = \sum_{j,m=1}^n \left( \frac{\partial \delta v_m^{*,\rho,l,k-1}}{\partial x_j} \frac{\partial v_j^{*,\rho,l,k-1}}{\partial x_m} \right) + \sum_{j,m=1}^n \left( \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \right) \quad (13) \\
& = \sum_{m,j=1}^n \left( \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} \right) + \sum_{j,m=1}^n \left( \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \right) \\
& = \sum_{m,j=1}^n \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} + \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \right).
\end{aligned}$$

#### 4.1 $L^2 \times L^2$ -estimates

We shall use Young's inequality. Let  $g \in L^1([a, a+1] \times \mathbb{R}^n)$  and  $f \in L^2([a, t] \times \mathbb{R}^n)$  for some  $a+1 \geq t > a \geq 0$ . Basically in our the Young inequality is the observation that

$$\begin{aligned}
& \left| \int_a^t \int_{\mathbb{R}^n} f(s, y) g(t-s, x-y) dy ds \right|_{L^2 \times L^2} \\
& = \left| \int_a^t \int_{\mathbb{R}^n} f(t-s, x-y) g(s, y) dy ds \right|_{L^2 \times L^2} \\
& \leq \left| \int_a^{a+1} \int_{\mathbb{R}^n} |f^{-s,-y}(t, x)| |g(s, y)| dy ds \right|_{L^2 \times L^2} \\
& \leq \|f\|_{L^2 \times L^2} \|g\|_{L^1 \times L^1} ds
\end{aligned} \quad (14)$$

where  $f^{-t,-y}(x) := f(x-y)$  denotes the function  $f$  shifted by  $-y$ , and we may use Minkowski's inequality. The functions considered are local with respect to time but we may consider  $s \rightarrow |f(s, \cdot)|_{L^p}$ ,  $t \rightarrow |g^{-s}(t, \cdot)|_{L^1}$  as functions in  $L^1(\mathbb{R})$  by defining them to be zero in  $\mathbb{R} \setminus [a, a+1]$ .

Now this type of Young inequalities may not be applied immediately in our situation, where one part of the convolution is a Gaussian. The application of Fourier transforms of fundamental solutions of the heat equation with respect to space *and* time variables shows this. For the partial (not normed) Fourier transformation with respect to the spatial variables we get for  $t > s$

$$\begin{aligned}
& \int_{\mathbb{R}^n} \exp(2\pi i \xi z) \left| \frac{1}{(2\sqrt{\epsilon\pi(t-s)})^n} \exp\left(-\frac{z^2}{4\epsilon(t-s)}\right) \right| dz \\
& = \exp(-4\epsilon(t-s)\pi^2 \xi^2).
\end{aligned} \quad (15)$$

Now as  $s \uparrow t$  this equation becomes 1 (the formal Fourier transform of the  $\delta$ -distribution), and this is not in  $L^1$ . However for the truncated fundamental solution

$$\phi_1(z)G_\epsilon(t-s, z) := \phi_1(z) \frac{1}{(2\sqrt{\epsilon\pi}(t-s))^n} \exp\left(-\frac{z^2}{4\epsilon(t-s)}\right) \quad (16)$$

the situation is different. Here  $\phi_1 \in C^\infty(B_1(0))$ , i.e., with support in  $B_1(0)$ , and with  $\phi_1(x) = 1$  for  $|x| \leq 0.5$ . Note that  $\phi_1$  and  $1 - \phi_1$  build a partition of unity on  $\mathbb{R}^n$ . So the idea for estimating the increments  $\delta v_i^{*, \rho, l, k}$  is to split up the integral of their representation and estimate one convolution summand with factor

$$G_\epsilon^B(t-s, z) := \phi_1(z)G_\epsilon(t-s, z) \quad (17)$$

via the Young inequality and the other convolution summand with factor

$$G_\epsilon^{(1-B)}(t-s, z) := (1 - \phi_1(z))G_\epsilon(t-s, z) \quad (18)$$

via a weighted convolution estimates for  $L^2$  functions. Indeed for the truncated Gaussian  $\phi_1(z)G_\epsilon(t-s, z)$  we have

$$\begin{aligned} |\phi_1(x-y)G_\epsilon(t-s, x-y)| &= \left| \phi_1(x-y) \frac{1}{(2\sqrt{\epsilon\pi}(t-s))^n} \exp\left(-\frac{(x-y)^2}{4\epsilon(t-s)}\right) \right| \\ &= \left| \phi_1(x-y)(t-s)^{m-n/2} \frac{1}{(x-y)^{2m}} \left(\frac{(x-y)^2}{4\epsilon(t-s)}\right)^m \frac{1}{(2\sqrt{\epsilon\pi})^n} \exp\left(-\frac{(x-y)^2}{4\epsilon(t-s)}\right) \right| \\ &\leq \left| \phi_1(x-y)(t-s)^{m-n/2} \frac{1}{(x-y)^{2m}} \left(\frac{(x-y)^2}{(t-s)}\right)^m \frac{1}{(4\epsilon)^m} \frac{1}{(2\sqrt{\epsilon\pi})^n} \exp\left(-\frac{(x-y)^2}{4\epsilon(t-s)}\right) \right| \\ &\leq \left| \phi_1(x-y)C(t-s)^{m-n/2} \frac{1}{(x-y)^{2m}} \right|, \end{aligned} \quad (19)$$

where

$$C := \sup_{z,t} \left| \frac{1}{(4\epsilon)^m} \frac{1}{(2\sqrt{\epsilon\pi})^n} \left(\frac{z^2}{t}\right)^m \exp\left(-\frac{z^2}{4\epsilon t}\right) \right| > 0 \quad (20)$$

is a constant depending on  $\epsilon$ , but finite for each  $\epsilon > 0$ . We may use this estimate locally for  $m = 1$ , i.e. we may use the upper bound

$$|\phi_1(x-y)G_\epsilon(t-s, x-y)| \leq |\phi_1(x-y)C(t-s)^{-1/2} \frac{1}{(x-y)^2}| \quad (21)$$

which is  $L^1$  for dimension  $n = 3$ , because of the localisation and may be used with mixed Young inequalities. We start with  $L^2$  estimates for

$\delta v_i^{*,\rho,l,k}(\tau, \cdot) = \delta v_i^{*,\rho,l,k}(\tau, \cdot)$  where we define

$$\begin{aligned}
\delta v_i^{*,\rho,l,k}(\tau, x) &:= \delta v_i^{*,\rho,l,k,1}(\tau, x) + \delta v_i^{*,\rho,l,k,2}(\tau, x) + \delta v_i^{*,\rho,l,k,3}(\tau, x) \\
&:= \rho_l \int_{l-1}^{\tau} \int_{\mathbb{R}^n} \left( - \sum_{j=1}^n \delta v_j^{*,\rho,l,k-1} \frac{\partial v_i^{*,\rho,l,k-1}}{\partial x_j} \right) (s, y) G_l(\tau - s; x - y) dy ds \\
&\quad + \rho_l \int_{l-1}^{\tau} \int_{\mathbb{R}^n} \left( - \sum_{j=1}^n v_j^{*,\rho,l,k-1} \frac{\partial \delta v_i^{*,\rho,l,k-1}}{\partial x_j} \right) (s, y) G_l(\tau - s; x - y) dy ds \\
&\quad + \rho_l \int_{l-1}^{\tau} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(z - y) \right) \times \\
&\quad \times \left( \sum_{m,j=1}^n \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} + \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \right) \right) (s, y) G_l(\tau - s, x - z) dy dz ds.
\end{aligned} \tag{22}$$

We have a convolution with respect to time and with respect to the spatial variables. We do some estimates for the  $L^2 \times L^2$ -norm, or, for the squared  $|\cdot|_{L^2 \times L^2}$ -norm. We may consider  $\tau \rightarrow \delta v_{\epsilon i}^{*,\rho,l,k,1}(\tau, x) = 1_{[l-1, l]} \delta v_{\epsilon i}^{*,\rho,l,k,1}(\tau, x)$  as an  $L^1$  function on  $\mathbb{R}$  where  $1_{[l-1, l]}$  is the function which equals 1 on the interval  $[l - 1, l]$  and is 0 on  $\mathbb{R} \setminus [l - 1, l]$ . We implicitly assume this for convenience without changing the symbol of the function. We now apply a Young inequality to the first term on the right side of (22). We obtain

$$\begin{aligned}
&|\delta v_{Bi}^{*,\rho,l,k,1}(\tau, \cdot)|_{L^2 \times L^2}^2 := \\
&= \left| \rho_l \int_{l-1}^{\tau} \int_{\mathbb{R}^n} \left( - \sum_{j=1}^n \delta v_j^{*,\rho,l,k-1} \frac{\partial v_i^{*,\rho,l,k-1}}{\partial x_j} \right) (s, y) G_l^B(\tau - s; \cdot - y) dy ds \right|_{L^2 \times L^2}^2 \\
&\leq \left| \rho_l \int_{l-1}^{\tau} \int_{\mathbb{R}^n} \left( \sum_{j=1}^n \delta v_j^{*,\rho,l,k-1} \frac{\partial v_i^{*,\rho,l,k-1}}{\partial x_j} \right) (s, y) \right| |G_l^B(\tau - s; \cdot - y)| dy ds \Big|_{L^2 \times L^2}^2 \\
&\leq \left| \rho_l C_{k-1} \int_{l-1}^{\tau} \int_{\mathbb{R}^n} \left( \sum_{j=1}^n |\delta v_j^{*,\rho,l,k-1}| \right) (s, y) \right| |G_l^B(\tau - s; \cdot - y)| dy ds \Big|_{L^2 \times L^2}^2 \\
&\leq \left| \left( \rho_l C_{k-1} n \max_{j \in \{1, \dots, n\}} \int_{l-1}^{\tau} \int_{\mathbb{R}^n} \left( |\delta v_j^{*,\rho,l,k-1}|(\tau - s, x - y) \right) |G_l^B(s, y)| dy ds \right) \right|_{L^2 \times L^2}^2 \\
&\leq \left| \rho_l C_{k-1} n \max_{j \in \{1, \dots, n\}} |\delta v_j^{*,\rho,l,k-1}|^{-s, -y}(t, x) \int_{\mathbb{R}} \int_{\mathbb{R}^n} |G_l^B(s, y)| dy ds \right|_{L^2 \times L^2}^2 \\
&\leq \rho_l^2 C_{k-1}^2 n^2 \max_{j \in \{1, \dots, n\}} |\delta v_j^{*,\rho,l,k-1}|_{L^2 \times L^2}^2 |G_l^B|_{L^1 \times L^1}^2 \\
&\leq \rho_l^2 C_{k-1}^2 n^2 (C_G^B)^2 \max_{j \in \{1, \dots, n\}} |\delta v_j^{*,\rho,l,k-1}|_{L^2 \times L^2}^2
\end{aligned} \tag{23}$$



where

$$C_{k-1} = \sup_{s \in [l-1, l], y \in \mathbb{R}^3} (1 + |y|^2) \sum_{0 \leq |\alpha| \leq 2} \left| D_x^\alpha v_i^{*, \rho, l, k-1}(s, y) \right|, \quad (24)$$

where  $D_x^\alpha$  denotes the multivariate partial derivative of order  $\alpha$  with multi-index  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ , and where

$$C_G^B := |G_l^B|_{L^1 \times L^1}. \quad (25)$$

Note that at this point the factor (24) seems obscure and we have could set it to 1 at this point. However, later we shall need weighted convolutions in  $L^2$  for the estimation of the Leray projection term and then we need this weight. The justification of the finiteness of  $C_{k-1}$  in (24), i.e., the polynomial decay of order  $m = 2$  will be justified in the next section. An analogous argument leads to

$$\begin{aligned} & \left| \delta v_{(1-B)i}^{*, \rho, l, k, 1} \right|_{L^2 \times L^2}^2 := \\ & = \left| \rho_l \int_{l-1}^\tau \int_{\mathbb{R}^n} \left( - \sum_{j=1}^n \delta v_j^{*, \rho, l, k-1} \frac{\partial v_i^{*, \rho, l, k-1}}{\partial x_j} \right) (s, y) G_l^{(1-B)}(\tau - s; \cdot - y) dy ds \right|_{L^2 \times L^2}^2 \\ & \leq \rho_l^2 C_{k-1}^2 n^2 (C_G^{1-B})^2 \max_{j \in \{1, \dots, n\}} \left| \delta v_j^{*, \rho, l, k-1}(\tau, \cdot) \right|_{L^2 \times L^2}^2, \end{aligned} \quad (26)$$

where

$$C_G^{1-B} := |G_l^{1-B}|_{L^1 \times L^1}. \quad (27)$$

Summing up our result for the first term on the right side of (22) we have (using  $(C_G^B)^2 + (C_G^{1-B})^2 \leq (C_G^B + C_G^{1-B})^2$ ) for all  $1 \leq i \leq n$

$$\begin{aligned} & \left| \delta v_i^{*, \rho, l, k, 1} \right|_{L^2 \times L^2}^2 \leq \\ & \leq \rho_l^2 C_{k-1}^2 n^2 \left( C_G^B + C_G^{1-B} \right)^2 \times \\ & \times \max_{j \in \{1, \dots, n\}} \left| \delta v_j^{*, \rho, l, k-1}(\tau, \cdot) \right|_{L^2}^2 |G_l|_{L^1 \times L^1} \\ & \leq \rho_l^2 C_{k-1}^2 n^2 C_G^2 \max_{j \in \{1, \dots, n\}} \left| \delta v_j^{*, \rho, l, k-1} \right|_{L^2 \times L^2}^2, \end{aligned} \quad (28)$$

where

$$C_G = C_G^B + C_G^{1-B}. \quad (29)$$

The latter estimate holds for all  $1 \leq i \leq n$ , hence we have

$$\begin{aligned} & \max_{j \in \{1, \dots, n\}} \left| \delta v_i^{*, \rho, l, k, 1} \right|_{L^2 \times L^2}^2 \leq \\ & \leq \rho_l^2 C_{k-1}^2 n^2 C_G^2 \max_{j \in \{1, \dots, n\}} \left| \delta v_j^{*, \rho, l, k-1}(\tau, \cdot) \right|_{L^2 \times L^2}^2. \end{aligned} \quad (30)$$

For the second term on the right side of (22), i.e., for  $\delta v_i^{*,\rho,l,k,2}$  we get the analogous estimate

$$\begin{aligned} \max_{j \in \{1, \dots, n\}} |\delta v_i^{*,\rho,l,k,2}(\tau, \cdot)|_{L^2}^2 &\leq \\ &\leq \rho_l^2 C_{k-1}^2 n^2 C_G^2 \max_{j \in \{1, \dots, n\}} \left| \frac{\partial}{\partial x_k} \delta v_j^{*,\rho,l,k-1}(\tau, \cdot) \right|_{L^2}^2, \end{aligned} \quad (31)$$

Finally we look at the third term on the right side of (22) which is a double convolution integrated over time. We do the estimate in the special case  $n = 3$  this time. Again we split up the function

$$\delta v_i^{*,\rho,l,k,3}(\tau, \cdot) = \delta v_{Bi}^{*,\rho,l,k,3}(\tau, \cdot) + \delta v_{(1-B)i}^{*,\rho,l,k,3}(\tau, \cdot), \quad (32)$$

corresponding to summands with a truncated Gaussian and its complement as above. Each of this summand is again split up into two summands one of which corresponds to a truncated Laplacian kernel  $\phi_1 K_{,i}$  (more precisely: its  $i$ th partial derivative), and the complement  $(1 - \phi_1) K_{,i}$ . We define

$$\delta v_{Bi}^{*,\rho,l,k,3}(\tau, \cdot) = \delta v_{BBi}^{*,\rho,l,k,3}(\tau, \cdot) + \delta v_{B(1-B)i}^{*,\rho,l,k,3}(\tau, \cdot) \quad (33)$$

and

$$\delta v_{(1-B)i}^{*,\rho,l,k,3}(\tau, \cdot) = \delta v_{(1-B)Bi}^{*,\rho,l,k,3}(\tau, \cdot) + \delta v_{(1-B)B(1-B)i}^{*,\rho,l,k,3}(\tau, \cdot), \quad (34)$$

where the references will be made precise in the following estimations. We start with the third term  $\delta v_{BBi}^{*,\rho,l,k,3}$  which is defined via

$$\begin{aligned} \delta v_{BBi}^{*,\rho,l,k,3}(\tau, \cdot) &= \rho_l \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \phi_1(z - y) \frac{\partial}{\partial x_i} K_n(z - y) \right) \times \\ &\times \left( \sum_{m,j=1}^n \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} + \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \right) \right) (s, y) \times \\ &\int_{l-1}^{\tau} G_l^B(\tau - s, x - z) dy dz \end{aligned} \quad (35)$$

This means that we have a truncated kernel  $\phi_1(\cdot) K_{,i}(\cdot)$  and a truncated Gaussian  $G_l^B$ . The double subscript  $B$  indicates that we have bounded support in both cases. Recall that we set  $\delta v_{BBi}^{*,\rho,l,k,3}(\tau, \cdot) = 0$  for  $\tau \in \mathbb{R} \setminus [l-1, l]$

in order to write the time integrals conveniently. We obtain

$$\begin{aligned}
& |\delta v_{BBi}^{*,\rho,l,k,3}|_{L^2 \times L^2}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\delta v_{BBi}^{*,\rho,l,k,3}(\tau, x)|^2 dx d\tau \\
& = \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\rho_l \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \phi_1(z-y) \frac{\partial}{\partial x_i} K_n(z-y) \right) \times \\
& \quad \times \left( \sum_{m,j=1}^n \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} + \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \right) \right) (s, y) \times \\
& \quad G_l^B(\tau-s, x-z) dy dz|^2 dx d\tau \\
& \leq \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\rho_l \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \left( \phi_1(z-y) \frac{\partial}{\partial x_i} K_n(z-y) \right) \times \right. \\
& \quad \times \left. \left( \sum_{m,j=1}^n \left| \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \right| \left| \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} + \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \right| \right) (s, y) \right| dy| \times \\
& \quad |G_l^{B,-z}(\tau-s, x-z)|^2 dz|^2 dx d\tau \\
& \leq \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\rho_l \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \phi_1(x-(z-y)) \frac{\partial}{\partial x_i} K_n(x-(z-y)) \right) \times \\
& \quad \times \left( \sum_{m,j=1}^n \left| \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \right| 2C_{k-1} \right) (\tau-s, y)| \times \\
& \quad |G^{B_l}(s, z)|^2 dy dz|^2 dx d\tau
\end{aligned} \tag{36}$$

where we recall our introduction of a function  $\phi_1 \in C^\infty(B_1(0))$ , i.e., with support in  $B_1(0)$ , and with  $\phi_1(x) = 1$  for  $|x| \leq 0.5$ . Note that  $\phi_1$  and  $1 - \phi_1$  build a partition of unity on  $\mathbb{R}^n$ . Furthermore we used our inductive assumption

$$\sup_{s \in [l-1, l], \sup_{x \in \mathbb{R}^3}} \left| \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} + \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \right| \leq C_{k-1} \tag{37}$$

The estimate (36) is still not suitable since the function  $\phi_1(\cdot) \frac{\partial}{\partial x_i} K_n(\cdot)$  is in  $L^1$  and not in  $L^2$ . For this reason, we have to shift the  $x$ -argument to the

function  $\frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m}$ . We observe that the right side of (36) can be written as

$$\begin{aligned}
& \left| \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\rho_l \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \phi_1^{-z+2y}(x-y) \frac{\partial}{\partial x_i} K_n^{-z+2y}(x-y) \right) \times \right. \\
& \times \left( \sum_{m,j=1}^n \left| \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \right| 2C_{k-1} \right) (\tau-s, y) \Big| \times \\
& |G^{B_l}(s, z)| dy dz \Big|^2 dx d\tau \\
& = \left| \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\rho_l \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \phi_1^{-z+2y}(y) \frac{\partial}{\partial x_i} K_n^{-z+2y}(y) \right) \times \right. \\
& \times \left( \sum_{m,j=1}^n \left| \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \right| 2C_{k-1} \right) (\tau-s, x-y) \Big| \times \\
& |G^{B_l}(s, z)| dy dz \Big|^2 dx d\tau.
\end{aligned} \tag{38}$$

Next the function

$$y \rightarrow \phi_1^{-z+2y}(y) \frac{\partial}{\partial x_i} K_n^{-z+2y}(y) \tag{39}$$

is certainly in  $L^1$  for  $n = 3$ , so we can apply the Young inequality again. Hence, from (36) we get

$$\begin{aligned}
& |\delta v_{BBi}^{*,\rho,l,k,3}(\tau, \cdot)|_{L^2 \times L^2}^2 \\
& \rho_l^2 4C_{k-1}^2 (C_G^B)^2 C_{K_3\phi_1}^2 \max_{m,j \in \{1, \dots, n\}} \left| \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \right|_{L^2 \times L^2}^2,
\end{aligned} \tag{40}$$

where

$$C_G^B = |G_l^B|_{L^1 \times L^1} \tag{41}$$

as before, and where we use the same observation on mixed forms of generalized Young inequalities as above. Furthermore we used that for  $n = 3$  we have

$$\left| \phi_{B_1(0)} \frac{\partial}{\partial x_i} K_n(\cdot) \right|_{L^1} \leq C_{K_3\phi_1} \tag{42}$$

for a finite constant  $C_{K_3\phi_1}$ .

Next for the second summand of  $\delta v_i^{*,\rho,l,k,3}$ , i.e., for the summand of the

form  $\delta v_{B(1-B)i}^{*,\rho,l,k,3}$  we have by an analogous argument

$$\begin{aligned}
& |\delta v_{B(1-B)i}^{*,\rho,l,k,3}|_{L^2 \times L^2}^2 \\
& \leq \rho_l^2 4C_{k-1}^2 \left| \int_{\mathbb{R}^n} \left( \phi_1^{-z+2y}(\cdot - y) \frac{\partial}{\partial x_i} K_n^{-z+2y}(\cdot - y) \right) \times \right. \\
& \quad \times \left( \sum_{m,j=1}^n \left| \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \right| \right)^{-s} (\tau, y) dy \Big|_{L^2 \times L^2}^2 |G_l^{(1-B)}|_{L^1 \times L^1} \\
& \leq \rho_l^2 4C_{k-1}^2 (C_G^{(1-B)})^2 \left| \int_{\mathbb{R}^n} \left( \phi_1^{-z+2y}(y) \frac{\partial}{\partial x_i} K_n^{-z+2y}(y) \right) \times \right. \\
& \quad \times \left( \sum_{m,j=1}^n \left| \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \right| \right) (\tau, \cdot - y) dy \Big|_{L^2}^2 \\
& \leq \rho_l^2 4C_{k-1}^2 (C_G^{(1-B)})^2 n^2 C_{K_3 \phi_1}^2 \max_{j,m \in \{1, \dots, n\}} \left| \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m}(\tau, \cdot) \right|_{L^2}^2
\end{aligned} \tag{43}$$

For the other two summands of  $\delta v_i^{*,\rho,l,k,3}$ , i.e., for the summand of the form  $\delta v_{(1-B)Bi}^{*,\rho,l,k,3}$  and  $\delta v_{(1-B)(1-B)i}^{*,\rho,l,k,3}$  we have to estimate kernels of the form

$$(1 - \phi_1)K_{,i} \tag{44}$$

while the treatment of the convolution with the Gaussian maintains, i.e., from the argument above we get

$$\begin{aligned}
& |\delta v_{(1-B)Bi}^{*,\rho,l,k,3}|_{L^2 \times L^2}^2 + |\delta v_{(1-B)(1-B)i}^{*,\rho,l,k,3}|_{L^2 \times L^2}^2 \\
& \leq \rho_l^2 4C_{k-1}^2 C_G^2 \left| \int_{\mathbb{R}^n} \left| (1 - \phi_1)^{-z+2y}(\cdot - y) \frac{\partial}{\partial x_i} K_n^{-z+2y}(\cdot - y) \right| \times \right. \\
& \quad \times \left( \sum_{m,j=1}^n \left| \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \right| \right)^{-y} (\tau, x) dy \Big|_{L^2}^2 \\
& \leq \rho_l^2 4C_{k-1}^2 C_G^2 n^2 \max_{j,m \in \{1, \dots, n\}} \left| \int_{\mathbb{R}^n} \left| (1 - \phi_1)^{-z+2y}(y) \times \right. \right. \\
& \quad \times \frac{\partial}{\partial x_i} K_n^{-z+2y}(y) \left| \left| \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \right|^{-y} (\tau, x) dy \right|_{L^2}^2 \\
& = \rho_l^2 4C_{k-1}^2 C_G^2 n^2 \max_{j,m \in \{1, \dots, n\}} \left| \int_{\mathbb{R}^n} \left| (1 - \phi_1)^{-z+2y}(y) \times \right. \right. \\
& \quad \times \frac{\partial}{\partial x_i} K_n^{-z+2y}(y) \left| \frac{1}{1+|y|^2} (1 + |y|^2) \left| \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \right|^{-y} (\tau, x) dy \right|_{L^2}^2
\end{aligned} \tag{45}$$

Now we do *not* have  $(1 - \phi_1)(\cdot) \frac{\partial}{\partial x_i} K_n(\cdot) \in L^1$ , and for this reason we introduced the constant  $C_{k-1}$  in the form (24) above. This means that we

can give another factor  $\frac{1}{1+|y|^2}$  to the convolution, i.e., we have

$$\begin{aligned}
& |\delta v_{(1-B)Bi}^{*,\rho,l,k,3}|_{L^2 \times L^2}^2 + |\delta v_{(1-B)(1-B)i}^{*,\rho,l,k,3}|_{L^2 \times L^2}^2 \\
& \leq \rho_l^2 4C_{k-1}^2 C_G^2 n^2 \max_{j,m \in \{1, \dots, n\}} \left| \int_{\mathbb{R}^n} |(1-\phi_1)^{-z+2y}(y) \times \right. \\
& \quad \left. \times \frac{\partial}{\partial x_i} K_n^{-z+2y}(y) \left| \frac{1}{1+|y|^2} \right| \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \right|^{-y}(\tau, x) dy \Big|_{L^2}^2
\end{aligned} \tag{46}$$

with the same constant  $C_{k-1}$ . Now we may use the fact that a function  $u$  is in  $L^2$  if  $s > \frac{n}{2}$ , and for all  $x \in \mathbb{R}^n$

$$u(x) := \int (1+|y|^2)^{-s/2} v(x-y) w(y) dy \tag{47}$$

for functions  $v, w \in L^2$ , and such that

$$|u|_{L^2} \leq C_s |v|_{L^2} |w|_{L^2}. \tag{48}$$

for a constant  $C_s > 0$ . Indeed, we have

$$(1-\phi_1)(y) \frac{\partial}{\partial x_i} K_n(y) \in L^2, \tag{49}$$

for  $n \geq 3$  where we may denote an  $L^2$ -bound by  $K_{3L^2}$ , and

$$y \rightarrow \left| \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \right|(\tau, y) \in L^2. \tag{50}$$

Hence, from (46) we obtain

$$\begin{aligned}
& |\delta v_{(1-B)i}^{*,\rho,l,k,3}|_{L^2 \times L^2}^2 = |\delta v_{(1-B)Bi}^{*,\rho,l,k,3}|_{L^2 \times L^2}^2 + |\delta v_{(1-B)(1-B)i}^{*,\rho,l,k,3}|_{L^2 \times L^2}^2 \\
& \leq \rho_l^2 4C_{k-1}^2 C_G^2 C_{K_3L^2}^2 n^2 C_s^2 \max_{j,m \in \{1, \dots, n\}} \left| \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \right|(\tau, \cdot) \Big|_{L^2}^2
\end{aligned} \tag{51}$$

Summing up and recalling that we have first order derivatives on the right side for some summands we write a  $H^1$  norm on the right side (which suffices for our purposes). We have

$$\begin{aligned}
& |\delta v_i^{*,\rho,l,k}|_{L^2 \times L^2}^2 \leq \rho_l^2 4C_{k-1}^2 C_G^2 n^2 \times \\
& \times (1 + (C_{K_3\phi_1} + C_{K_3L^2})^2 (1 + C_s^2)) \max_{j \in \{1, \dots, n\}} \left| \delta v_j^{*,\rho,l,k-1} \right|_{L^2 \times H^1}^2.
\end{aligned} \tag{52}$$

We note that for the old scheme considered in [4] and [1] we would have an estimate with right side in  $L^2$  which is otherwise the same as in (52) up to a constant related to the fact that we have one source term less and that the Gaussian estimates for the local fundamental solutions with variable drift term and their adjoints produce different constants.

## 4.2 $L^2 \times H^1$ estimates and $L^2 \times H^2$ -estimates

In addition to the  $L^2$ -estimates we need estimates for the first order partial derivatives of the components of the value function. We start with estimates for norms which include spatial derivatives, i.e., we start with  $L^2 \times H^1$ -estimates. At each stage  $k$  of the construction we may differentiate under the integral and start with the pointwise valid expression

$$\begin{aligned}
& \frac{\partial}{\partial x_j} \delta v_i^{*,\rho,l,k}(\tau, x) \\
&=: \frac{\partial}{\partial x_j} \delta v_i^{*,\rho,l,k,1}(\tau, x) + \frac{\partial}{\partial x_j} \delta v_{\epsilon i}^{*,\rho,l,k,2}(\tau, x) + \frac{\partial}{\partial x_j} \delta v_i^{*,\rho,l,k,3}(\tau, x) \\
&:= \rho_l \int_{l-1}^\tau \int_{\mathbb{R}^n} \left( - \sum_{j=1}^n \delta v_j^{*,\rho,l,k-1} \frac{\partial v_i^{*,\rho,l,k-1}}{\partial x_j} \right) (s, y) G_{l,j}(\tau - s; x - y) dy ds \\
&+ \rho_l \int_{l-1}^\tau \int_{\mathbb{R}^n} \left( - \sum_{j=1}^n v_j^{*,\rho,l,k-1} \frac{\partial \delta v_i^{*,\rho,l,k-1}}{\partial x_j} \right) (s, y) G_{l,j}(\tau - s; x - y) dy ds \\
&+ \rho_l \int_{l-1}^\tau \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(z - y) \right) \times \\
&\times \left( \sum_{m,j=1}^n \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} + \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \right) \right) (s, y) G_{l,j}(\tau - s, x - z) dy dz ds,
\end{aligned} \tag{53}$$

where the subscript  $_{,j}$  denotes partial derivatives with respect to the  $j$ th spatial variable (as usual in Einstein notation). From the representation in (53) we observe that the argument for the  $L^2$  estimates can be repeated, if we have a  $L^1 \times L^1$ -bound for the first order partial derivatives of the Gaussian  $G_l$  (first order partial derivatives with respect to the spatial variables). We have to refine the simple estimate in (19) a bit, observing that for  $\alpha \in (1.5, 2)$  and  $n = 3$  we have

$$\begin{aligned}
& |G_{\epsilon,i}(t - s, x - y)| = \left| \frac{1}{(2\sqrt{\epsilon\pi}(t-s))^n} \frac{-(x_i - y_i)}{2(t-s)} \exp\left(-\frac{(x-y)^2}{4\epsilon(t-s)}\right) \right| \\
&\leq |(t-s)^{\alpha-1-n/2} \frac{1}{|x-y|^{2\alpha}} \left( \frac{(x-y)^2}{4\epsilon(t-s)} \right)^\alpha \frac{1}{(2\sqrt{\epsilon\pi})^n} \exp\left(-\frac{(x-y)^2}{4\epsilon(t-s)}\right)| \\
&\leq |(t-s)^{\alpha-1-n/2} \frac{1}{|x-y|^{2\alpha-1}} \left( \frac{(x-y)^2}{(t-s)} \right)^m \frac{1}{(4\epsilon)^\alpha} \frac{1}{(2\sqrt{\epsilon\pi})^n} \exp\left(-\frac{(x-y)^2}{4\epsilon(t-s)}\right)| \\
&\leq |C(t-s)^{\alpha-1-n/2} \frac{1}{|x-y|^{2\alpha-1}}|.
\end{aligned} \tag{54}$$

It follows that we have local integrability with respect to time, since  $\alpha - 1 - n/2 \in (-1, 0)$  and in space since  $\frac{1}{|x-y|^{2\alpha-1}}$  is locally integrable in dimension  $n = 3$  for  $2\alpha - 1 \in (2, 3)$ . Note that we have the same constant  $C$  as in the

simple estimate (19) above. Hence, we have

$$|\phi_1(x-y)G_{\epsilon,i}(t-s, x-y)|_{L^1 \times L^1} \leq C_G^{B1}, \quad (55)$$

and it is clear that

$$|(1-\phi_1)(x-y)G_{\epsilon,i}(t-s, x-y)|_{L^1 \times L^1} \leq C_G^{(1-B)1}. \quad (56)$$

Hence we may apply the same arguments as for  $L^2 \times$ -estimates where we have to replace the constants  $C_G$  for the Gaussian by  $C_G^1 = C_G^{B1} + C_G^{1(1-B)}$ , and get

$$\begin{aligned} |\delta v_i^{*,\rho,l,k}|_{L^2 \times H^1}^2 &= \sum_{j=1}^3 \left( |\delta v_{Bi}^{*,\rho,l,k,j}|_{L^2 \times H^1}^2 + |\delta v_{(1-B)i}^{*,\rho,l,k,j}|_{L^2 \times H^1}^2 \right) \\ &\leq (n+1)\rho_l^2 4C_{k-1}^2 (C_G^1)^2 n^2 (1 + (C_{K_{3\phi_1}} + K_{3L^1})^2 (1 + C_s^2)) \times \\ &\quad \times \max_{j \in \{1, \dots, n\}} |\delta v_j^{*,\rho,l,k-1}|_{L^2 \times H^1}^2 \end{aligned} \quad (57)$$

Note that the additional factor  $n+1$  takes account of the fact that we need to estimate  $n+1 = 4$  terms with the method of  $L^2$ -estimates above. For  $L^2 \times H^2$ -estimates we use convolution rules and partial integration to get the following representation of the second order partial derivatives of the value function. We have

$$\begin{aligned} &\frac{\partial^2}{\partial x_m \partial x_j} \delta v_i^{*,\rho,l,k}(\tau, x) \\ &=: \frac{\partial^2}{\partial x_m \partial x_j} \delta v_i^{*,\rho,l,k,1}(\tau, x) + \frac{\partial^2}{\partial x_m \partial x_j} \delta v_{\epsilon i}^{*,\rho,l,k,2}(\tau, x) + \frac{\partial^2}{\partial x_m \partial x_j} \delta v_i^{*,\rho,l,k,3}(\tau, x) \\ &:= \rho_l \int_{l-1}^\tau \int_{\mathbb{R}^n} \left( - \sum_{j=1}^n \delta v_j^{*,\rho,l,k-1} \frac{\partial v_i^{*,\rho,l,k-1}}{\partial x_j} \right)_{,j} (s, y) G_{l,m}(\tau-s; x-y) dy ds \\ &\quad + \rho_l \int_{l-1}^\tau \int_{\mathbb{R}^n} \left( - \sum_{j=1}^n v_j^{*,\rho,l,k-1} \frac{\partial \delta v_i^{*,\rho,l,k-1}}{\partial x_j} \right)_{,j} (s, y) G_{l,m}(\tau-s; x-y) dy ds \\ &\quad + \rho_l \int_{l-1}^\tau \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(z-y) \right) \times \\ &\quad \times \left( \sum_{m,j=1}^n \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} + \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \right) \right)_{,j} (s, y) \times \\ &\quad \times G_{l,m}(\tau-s, x-z) dy dz ds. \end{aligned} \quad (58)$$

Here it becomes clear why we included second derivatives in the definition of  $C_{k-1}$ . Proceeding as before we need to apply the product rule in order to expand the derivatives  $_{,j}$  of the value functions above. This gives an



additional factor 2 at  $C_{k-1}$ . Furthermore we have  $1 + n + n^2$  terms that we have to estimate. Hence,

$$\begin{aligned} |\delta v_i^{*,\rho,l,k}|_{L^2 \times H^2}^2 &= \sum_{j=1}^3 \left( |\delta v_{Bi}^{*,\rho,l,k,j}|_{L^2 \times H^2}^2 + |\delta v_{(1-B)i}^{*,\rho,l,k,j}|_{L^2 \times H^2}^2 \right) \\ &\leq (n^2 + n + 1) \rho_l^2 4C_{k-1}^2 (C_G^1)^2 n^2 (1 + (C_{K_3\phi_1} + C_{K_3L^1})^2 (1 + C_s^2)) \times \\ &\quad \times \max_{j \in \{1, \dots, n\}} |\delta v_j^{*,\rho,l,k-1}(\tau, \cdot)|_{L^2 \times H^2}^2 \end{aligned} \quad (59)$$

### 4.3 Proof of preservation of polynomial decay of order $m$ , and higher order estimates

In the representation of the functions  $v_i^{*,\rho,k,l}$ ,  $1 \leq i \leq n$  as in (12) we have three summands

$$\begin{aligned} &\rho_l \int_{l-1}^\tau \int_{\mathbb{R}^n} \left( - \sum_{j=1}^n \delta v_j^{*,\rho,l,k-1} \frac{\partial v_i^{*,\rho,l,k-1}}{\partial x_j} \right) (s, y) G_l(\tau - s; x - y) dy ds, \\ &\rho_l \int_{l-1}^\tau \int_{\mathbb{R}^n} \left( - \sum_{j=1}^n v_j^{*,\rho,l,k-1} \frac{\partial \delta v_i^{*,\rho,l,k-1}}{\partial x_j} \right) (s, y) G_l(\tau - s; x - y) dy ds, \\ &\rho_l \int_{l-1}^\tau \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(z - y) \right) \times \\ &\quad \times \left( \sum_{m,j=1}^n \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} + \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \right) \right) (s, y) \times \\ &\quad \times G_l(\tau - s, x - z) dy dz ds. \end{aligned} \quad (60)$$

All these summands have products of functions and derivatives of functions of the form  $\delta v_j^{*,\rho,l,k-1}$ ,  $v_j^{*,\rho,l,k-1}$  known from the previous iteration step. Similar is true for representations of the functions  $v_i^{*,\rho,k,l}$ ,  $1 \leq i \leq n$ . Hence if these functions  $\delta v_j^{*,\rho,l,k-1}$ ,  $v_j^{*,\rho,l,k-1}$ , and derivatives of these functions say of order up to  $|\alpha| \leq m$  have polynomial decay of order  $p$  then products have polynomial decay of order  $2p$ . This order of polynomial decay may be weakened by the convolution with the Gaussian or by the convolution with the Laplacian kernel in the Leray projection term, but we may expect that for  $p > n$  the polynomial decay of order  $p$  may be preserved. In the next lemma we analyze this.

**Lemma 4.1.** *Let  $\tau \in [l-1, l]$  for some time step number  $l \geq 1$ . Polynomial decay of order  $m \geq 2$  and existence of continuous derivatives up to order  $m$  is preserved by the local scheme  $v_i^{*,\rho,k,l}$ ,  $1 \leq i \leq n$  if polynomial decay of order  $m$  is given for the functions  $v^{*,\rho,l,k}(\tau, \cdot) \in C^m \cap H^m$ , and  $\delta v^{*,\rho,l,k}(\tau, \cdot) \in C^m \cap H^m$ . Especially,  $v^{*,\rho,l,k+1}(\tau, \cdot) \in C^m \cap H^m$ , and  $\delta v^{*,\rho,l,k+1}(\tau, \cdot) \in C^m \cap H^m$ .*

*Proof.* We show that for  $m \geq 1$  and  $0 \leq |\alpha| \leq 2$  we have

$$|D_x^\alpha \delta v^{*,\rho,l,k+1}| \leq \frac{1}{|x|^m}, \text{ if } |x| \geq 1 \quad (61)$$

if this holds for  $|D_x^\alpha \delta v^{*,\rho,l,k}|$ . Similarly for higher order derivatives  $D_x^\alpha v^{*,\rho,l,k+1}$  with  $|\alpha| > 2$  and some  $\rho_l > 0$ , where  $D_x^\alpha = D_x^{\alpha_1} D_x^{\alpha_2} \dots D_x^{\alpha_n}$  denotes the multivariate partial derivative with respect to the multiindex  $\alpha = (\alpha_1, \dots, \alpha_n)$ . We have to estimate convolutions with the Gaussian  $G_\epsilon$  for some small  $\epsilon > 0$ . The expressions for  $\delta v_i^{*,\rho_l,k+1,l,k+1}$  and  $v_i^{\epsilon,k+1}$  and their derivatives (cf. (58) for second order spatial derivatives; higher order spatial derivatives can be represented similarly with first order spatial derivatives of the Gaussian) involve terms which are essentially of the form

$$\int_{l-1}^\tau \int_{\mathbb{R}^3} h(y) G_\epsilon(t-s, x-y) dy ds, \quad (62)$$

or

$$\int_{l-1}^\tau \int_{\mathbb{R}^3} h(y) G_{\epsilon,j}(t-s, x-y) dy ds, \quad (63)$$

where  $h$  is some function which is a functional of  $D_x^\beta v^{*,\rho,l,k}(s, \cdot)$  and  $D_x^\beta \delta v^{*,\rho,l,k}(s, \cdot)$  for  $0 \leq |\beta| \leq m$ , and where the latter functions are in  $C^m \cap H^m$  and such that the functions themselves and their derivatives up to order  $m$  are assumed to be of polynomial decay of order  $m$  (according to inductive assumption). Furthermore we have a Gaussian factor of the form  $G_\epsilon(t-s, x-y)$  (defined analogously as  $G_l$  above), or first order partial derivatives of this factor. We split up the integral of the convolution into two parts where one part is the integral for  $|y| \leq \frac{|x|}{2}$ . On this domain we observe that the Gaussian has polynomial decay of any order. Indeed we have for  $0 < |t-s| \leq 1$ ,  $m \geq \frac{n}{2}$ , and  $|y| \leq \frac{|x|}{2} > 0$

$$\begin{aligned} |G_\epsilon(t-s, x-y)| &= \left| \frac{1}{(2\sqrt{\epsilon\pi(t-s)})^n} \exp\left(-\frac{(x-y)^2}{4\epsilon(t-s)}\right) \right| \\ &= \left| (t-s)^{m-n/2} \frac{1}{(x-y)^{2m}} \left(\frac{(x-y)^2}{(t-s)}\right)^m \frac{1}{(2\sqrt{\epsilon\pi})^n} \exp\left(-\frac{(x-y)^2}{4\epsilon(t-s)}\right) \right| \\ &\leq \left| C(t-s)^{m-n/2} \frac{1}{(x-y)^{2m}} \right| \leq \frac{C'}{|x|^{2m}}, \end{aligned} \quad (64)$$

where  $C' = cC$ , where with  $z = \frac{(x-y)}{\sqrt{t-s}}$  we define

$$C := \sup_{z>0} \left| \frac{1}{(2\sqrt{\epsilon\pi})^n} (z^2)^m \exp\left(-\frac{z^2}{4\epsilon}\right) \right| > 0 \quad (65)$$

is a finite constant and  $c > 0$  is another finite constant. Furthermore, for the first order partial derivatives of the Gaussian we have a similar estimate,

i.e., we have for  $1 \leq i \leq n$ , and  $0 < |t - s| \leq 1$ ,  $m > \frac{n}{2}$ , and  $|y| \leq \frac{|x|}{2} > 0$

$$\begin{aligned}
|G_{\epsilon,i}(t-s, x-y)| &= \left| \frac{1}{(2\sqrt{\epsilon\pi}(t-s))^n} \frac{-(x-y)_i}{2\epsilon(t-s)} \exp\left(-\frac{(x-y)^2}{4\epsilon(t-s)}\right) \right| \\
&= |(t-s)^{m-n/2-1} \frac{|x-y|}{(x-y)^{2m}} \left(\frac{(x-y)^2}{(t-s)}\right)^m \frac{1}{2\epsilon(2\sqrt{\epsilon\pi})^n} \exp\left(-\frac{(x-y)^2}{4\epsilon(t-s)}\right)| \\
&\leq |(t-s)^{m-n/2-1} \frac{1}{|x-y|^{2m-1}} \left(\frac{(x-y)^2}{(t-s)}\right)^m \frac{1}{(2\sqrt{\epsilon\pi})^n} \exp\left(-\frac{(x-y)^2}{4\epsilon(t-s)}\right)| \\
&\leq |C(t-s)^{m-n/2-1} \frac{1}{|x-y|^{2m-1}}| \leq (t-s)^{m-n/2-1} \frac{C'}{|x|^{2m-1}},
\end{aligned} \tag{66}$$

for some constant  $C' > 0$ , and with a locally integrable time factor which becomes nonsingular for  $m \geq 3$  in the case of dimension  $n = 3$ . On the complementary domain  $|y| > \frac{|x|}{2}$  we need some properties of the integrand  $h$ . We observe for  $|x| > 0$  and some constants  $C, C' > 0$

$$\begin{aligned}
&\int_{l-1}^{\tau} \int_{\{|y| \geq \frac{|x|}{2}\}} \frac{C}{y^{2p}} G_{\epsilon}(t-s, x-y) dy ds \\
&\leq \int_{l-1}^{\tau} \int_{\{|y| \geq \frac{|x|}{2}\} \& \{|x-y| \leq 1\}} \frac{C}{y^{2p}} G_{\epsilon}(t-s, x-y) dy ds \\
&+ \int_{l-1}^{\tau} \int_{\{|y| \geq \frac{|x|}{2}\} \& \{|x-y| > 1\}} \frac{C}{y^{2p}} G_{\epsilon}(t-s, x-y) dy ds \\
&\leq \int_{l-1}^{\tau} \int_{\{|y| \geq \frac{|x|}{2}\} \& \{|x-y| \leq 1\}} \frac{C}{y^{2p}} |C(t-s)^{-1/2} \frac{1}{(x-y)^2}| dy ds \\
&+ \int_{l-1}^{\tau} \int_{\{|y| \geq \frac{|x|}{2}\} \& \{|x-y| > 1\}} \frac{C}{y^{2p}} dy ds \leq \frac{C'}{|x|^{2p-n}}.
\end{aligned} \tag{67}$$

Note if the functions  $v_i^{*,\rho,l,k}, \delta v_i^{*,\rho,l,k}$  have polynomial decay of order  $m$  then all terms in the source function  $h$  except that Leray projection term have polynomial decay of order  $2m$  hence for  $m > n$  the preceding observation indicates that the polynomial decay might be preserved. Hence the proof reduces to the observation that the integrand  $h$  in the form it has in the representation of  $\delta v_i^{k+1}$  and their derivatives is of polynomial decay of order larger than  $p+n$ . Let us look at arbitrary partial derivatives of some maximal order  $m$  which are assumed to be of polynomial decay of order  $m$  inductively. Now from (22) we get for each  $1 \leq j \leq n$  and  $\alpha = (\alpha_1, \dots, \alpha_i, \dots, \alpha_n) =:$

$\beta + 1_j := (\alpha_1, \dots, \beta_j + 1, \dots, \alpha_n)$  the representation

$$\begin{aligned}
& D_x^\alpha \delta v_i^{*,\rho,l,k}(\tau, x) \\
& =: D_x^\alpha \delta v_i^{*,\rho,l,k,1}(\tau, x) + D_x^\alpha \delta v_i^{*,\rho,l,k,2}(\tau, x) + D_x^\alpha \delta v_i^{*,\rho,l,k,3}(\tau, x) \\
& := \rho_l \int_{l-1}^\tau \int_{\mathbb{R}^n} \left( - \sum_{j=1}^n \delta v_j^{*,\rho,l,k-1} \frac{\partial v_i^{*,\rho,l,k-1}}{\partial x_j} \right)_{,\beta} (s, y) G_{l,j}(\tau - s; x - y) dy ds \\
& + \rho_l \int_{l-1}^\tau \int_{\mathbb{R}^n} \left( - \sum_{j=1}^n v_j^{*,\rho,l,k-1} \frac{\partial \delta v_i^{*,\rho,l,k-1}}{\partial x_j} \right)_{,\beta} (s, y) G_{l,j}(\tau - s; x - y) dy ds \\
& + \rho_l \int_{l-1}^\tau \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(z - y) \right) \times \\
& \times \left( \sum_{m,j=1}^n \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} + \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \right) \right)_{,\beta} (s, y) \times \\
& \times G_{l,j}(\tau - s, x - z) dy dz ds,
\end{aligned} \tag{68}$$

where the subscript  $_{,\beta}$  denotes multivariate partial derivatives with respect to the multiindex  $\beta$ . Now concerning the first two terms in (68), i.e.,  $D_x^\alpha \delta v_i^{*,\rho,l,k,1}(\tau, x)$  and  $D_x^\alpha \delta v_i^{*,\rho,l,k,2}(\tau, x)$  the respective integrands

$$\left( - \sum_{j=1}^n \delta v_j^{*,\rho,l,k-1} \frac{\partial v_i^{*,\rho,l,k-1}}{\partial x_j} \right)_{,\beta}, \tag{69}$$

$$\left( - \sum_{j=1}^n v_j^{*,\rho,l,k-1} \frac{\partial \delta v_i^{*,\rho,l,k-1}}{\partial x_j} \right)_{,\beta} \tag{70}$$

are sums of products of functions  $D_x^\beta \delta v_j^{*,\rho,l,k-1}$  and  $D_x^\gamma v_j^{*,\rho,l,k-1}$  of order  $|\gamma|, |\beta| \leq m$ , which are assumed to be of polynomial decay of order  $m$ . Hence all the integrands except for the integrand related to the Leray projection term are of polynomial decay of order  $2m$ , i.e., the argument above concerning the estimate for polynomial decay of order  $m$  for convolutions with Gaussians shows that  $D_x^\alpha \delta v_i^{*,\rho,l,k,1}(\tau, x)$  and  $D_x^\alpha \delta v_i^{*,\rho,l,k,2}(\tau, x)$  are indeed of polynomial decay of order  $m$  for  $|\alpha| \leq m$ . It remains to check the polynomial decay of order  $m$  for the term  $D_x^\alpha \delta v_i^{*,\rho,l,k,3}(\tau, x)$  for  $|\alpha| \leq m$ . Note that for  $|y| \leq \frac{|x|}{2}$  we can use the Gaussian polynomial decay as above. Hence it is sufficient to estimate integrals of the form

$$\int_{\{|y| \geq \frac{|x|}{2}\}} K_{,i}(y - z) g(z) dz \tag{71}$$

where  $g$  is an integrand of the form

$$\left( \sum_{m,j=1}^n \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} + \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \right) \right)_{,\beta} \quad (72)$$

with  $|\beta| \leq m-1$ . Note that  $g$  is again a functional determined by sums of products of functions  $D_x^\beta \delta v_j^{*,\rho,l,k-1}$  and  $D_x^\gamma v_j^{*,\rho,l,k-1}$  of order  $|\gamma|, |\beta| \leq m$  such that  $g$  is of polynomial decay of order  $m$  by assumption, i.e., we have

$$|g(z)| \leq \frac{C}{z^{2m}} \text{ for } |z| \geq \frac{|x|}{4}. \quad (73)$$

We can use a similar argument as above and write

$$\begin{aligned} & \left| \int_{\{|y| \geq \frac{|x|}{2}\}} K_{,i}(y-z)g(z)dz \right| \\ & \leq \left| \int_{\{|y| \geq \frac{|x|}{2}\} \& \{|z| \leq \frac{|y|}{2}\}} K_{,i}(y-z)g(z)dz \right| \\ & + \left| \int_{\{|y| \geq \frac{|x|}{2}\} \& \{|z| > \frac{|y|}{2}\}} K_{,i}(y-z)g(z)dz \right| \\ & \leq \left| \int_{\{|y| \geq \frac{|x|}{2}\} \& \{|z| \leq \frac{|y|}{2}\}} \frac{\partial^{2m-2}}{\partial x_i^{2m-2}} K_{,i}(y-z) \frac{C}{z^2} dz \right| \\ & + \left| \int_{\{|y| \geq \frac{|x|}{2}\} \& \{|z| > \frac{|y|}{2}\}} K_{,i}(y-z) \frac{C}{y^{2m}} dz \right| \in O\left(\frac{C}{|x|^{2m-n}}\right) \end{aligned} \quad (74)$$

This shows that  $D_x^\alpha \delta v_i^{*,\rho,l,k,3}(\tau, x)$  for  $|\alpha| \leq m$  and  $m > n$  these functions are all of polynomial decay of order  $m$ , too.  $\square$

Next we consider the higher order estimates. Especially, we need the  $H^1 \times H^m$  estimate for some  $m$  and with  $H^1$  with respect to time. Product rules for Sobolev spaces allow us to reduce  $H^1 \times H^m$ -estimates to  $L^2 \times H^{m-2}$ -estimates. Next the estimates are considered in the case  $n = 3$ . First we observe that for all  $k \geq 0$  and  $m > \frac{5}{2}$  we have

$$\frac{\partial v_i^{\rho,l,k}}{\partial \tau} \in L^2 \times H^{m-2}, \quad (75)$$

and more generally for  $m > \frac{5}{2} + 2p$  we have

$$\frac{\partial^p v_i^{\rho,l,k}}{\partial \tau^p} \in L^2 \times H^{m-2p}. \quad (76)$$

In this context note that the local-time functions  $\frac{\partial^p v_i^{\rho,l,k}}{\partial \tau^p}$  are considered to be trivially extended to the whole time as mentioned above. Consider the

first equation of (2) and the case  $k = 1$ . We have the representation

$$\begin{aligned} \frac{\partial v_i^{\rho,l,k}}{\partial \tau} &= \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{\rho,l,k}}{\partial x_j^2} - \rho_l \sum_{j=1}^n v_j^{\rho,l,k-1} \frac{\partial v_i^{\rho,l,k}}{\partial x_j} \\ &+ \rho_l \sum_{j,m=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{m=1}^n \left( \frac{\partial v_m^{\rho,l,k-1}}{\partial x_j} \frac{\partial v_j^{\rho,l,k-1}}{\partial x_m} \right) (\tau, y) dy. \end{aligned} \quad (77)$$

Knowing that  $v_i^{\rho,l,k} \in L^2 \times H^m$  for the first term on the right side of (77) we have

$$\rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{\rho,l,k}}{\partial x_j^2} \in L^2 \times H^{m-2}. \quad (78)$$

Furthermore, for  $v_i^{\rho,l,k} \in L^2 \times H^m$  we observe that for  $m > \frac{5}{2}$  concerning the convection term on the right side of (77) we have for all  $\tau \in [l-1, l]$

$$- \rho_l \sum_{j=1}^n v_j^{\rho,l,k-1}(\tau, \cdot) \frac{\partial v_i^{\rho,l,k}}{\partial x_j}(\tau, \cdot) \in H^{m-1} \quad (79)$$

since the factors satisfy  $v_j^{\rho,l,k-1}(\tau, \cdot) \in H^m$  and  $\frac{\partial v_i^{\rho,l,k}}{\partial x_j}(\tau, \cdot) \in H^{m-1}$ , hence the product rule

$$|fg|_{H^s} \leq C_s |f|_{H^s} |g|_{H^s} \text{ for } f, g \in H^s, \quad s > \frac{n}{2} \quad (80)$$

applies for  $m > \frac{5}{2}$  in case  $n = 3$ . From our construction we know that the right side of (77) is locally continuous with respect to time. Hence, we have

$$- \rho_l \sum_{j=1}^n v_j^{\rho,l,k-1} \frac{\partial v_i^{\rho,l,k}}{\partial x_j} \in L^2 \times H^{m-1}. \quad (81)$$

Finally, concerning the Leray projection term in (77) we observe again that locally, i.e., for  $x - y \in B_r(0)$  for some ball  $B_r(0)$  of radius  $r > 0$  we have

$$K_{,i} \in L^1, \quad (82)$$

hence with an appropriate partition of unity, for example with  $\phi_1$  defined above we have

$$\phi_1 K_{,i} \in L^1, \quad \text{and } (1 - \phi_1) K_{,i} \in L^2 \quad (83)$$

for the first order partial derivatives of the Laplacian kernel. For  $m > \frac{5}{2}$  in case  $n = 3$  the product rule (80) can be applied to the (relevant part) of the integrand in the Leray projection term, and using the pointwise rule  $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$  in addition we conclude that

$$\sum_{j,m=1}^n \left( \frac{\partial v_m^{\rho,l,k-1}}{\partial x_j} \frac{\partial v_j^{\rho,l,k-1}}{\partial x_m} \right) (\tau, \cdot) \in L^1 \cap H^{m-1}. \quad (84)$$

Hence for all  $\tau \in [l-1, l]$

$$\begin{aligned}
& \rho_l \sum_{j,m=1}^n \int_{\mathbb{R}^n} \left( \phi_1(\cdot - y) \frac{\partial}{\partial x_i} K_n(\cdot - y) \right) \sum_{j,m=1}^n \left( \frac{\partial v_m^{\rho,l,k-1}}{\partial x_j} \frac{\partial v_j^{\rho,l,k-1}}{\partial x_m} \right) (\tau, y) dy \\
& + \rho_l \sum_{j,m=1}^n \int_{\mathbb{R}^n} \left( (1 - \phi_1(\cdot - y)) \frac{\partial}{\partial x_i} K_n(\cdot - y) \right) \times \\
& \times \sum_{j,m=1}^n \left( \frac{\partial v_m^{\rho,l,k-1}}{\partial x_j} \frac{\partial v_j^{\rho,l,k-1}}{\partial x_m} \right) (\tau, y) dy \in H^{m-1},
\end{aligned} \tag{85}$$

applying the product rule and different appropriate types of Young's inequality to both summands (cf. also part II of this investigation). Again continuity with respect to time leads to

$$\begin{aligned}
& \rho_l \sum_{j,m=1}^n \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} K_n(\cdot - y) \times \\
& \times \sum_{j,m=1}^n \left( \frac{\partial v_m^{\rho,l,k-1}}{\partial x_j} \frac{\partial v_j^{\rho,l,k-1}}{\partial x_m} \right) (\cdot, \cdot - y) dy \in L^2 \times H^{m-1}.
\end{aligned} \tag{86}$$

Hence, we have

$$\frac{\partial \delta v_i^{*,\rho,l,k}}{\partial \tau} \in L^2 \times H^{m-2} \tag{87}$$

for all  $k \geq 0$ . The next step is to show that we have a contraction for some  $\rho_l > 0$ . We observe

$$\begin{aligned}
& \left| \frac{\partial \delta v_i^{*,\rho,l,k}}{\partial \tau} \right|_{L^2 \times H^{m-2}} \leq \left| \frac{\partial \delta v_i^{*,\rho,l,k}}{\partial \tau} \right|_{L^2 \times H^m} \leq \rho_l \nu \sum_{j=1}^n \left| \frac{\partial^2 \delta v_i^{*,\rho,l,k}}{\partial x_j^2} \right|_{L^2 \times H^m} \\
& + \rho_l \sum_{j=1}^n \left| v_j^{*,\rho,l,k-1} \frac{\partial \delta v_i^{*,\rho,l,k-1}}{\partial x_j} \right|_{L^2 \times H^m} + \rho_l \sum_{j=1}^n \left| \delta v_j^{*,\rho,l,k-1} \frac{\partial v_i^{*,\rho,l,k-1}}{\partial x_j} \right|_{L^2 \times H^m} \\
& + \rho_l \left| \int_{l-1}^\tau \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(z - y) \right) \times \right. \\
& \times \left( \sum_{m,j=1}^n \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} + \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \right) \right) (s, y) \Big|_{L^2 \times H^m}
\end{aligned} \tag{88}$$

We can estimate the right side of (88) and for  $m > \frac{5}{2}$  we have the upper

bound

$$\begin{aligned}
& \rho_l \nu n \max_{j \in \{1, \dots, n\}} \left| \delta v_i^{*, \rho, l, k} \right|_{L^2 \times H^{m-2}} \\
& + \rho_l C_{5/2} n \max_{j \in \{1, \dots, n\}} \left| v_j^{*, \rho, l, k-1} \right|_{L^2 \times H^{m-1}} \left| \frac{\partial \delta v_i^{*, \rho, l, k-1}}{\partial x_j} \right|_{L^2 \times H^{m-1}} \\
& + \rho_l C_{5/2} n \max_{j \in \{1, \dots, n\}} \left| \delta v_j^{*, \rho, l, k-1} \right|_{L^2 \times H^{m-1}} \left| \frac{\partial v_i^{*, \rho, l, k-1}}{\partial x_j} \right|_{L^2 \times H^{m-1}} \quad (89) \\
& + \rho_l C_{5/2} C_K n^2 \max_{j, m \in \{1, \dots, n\}} \left| \frac{\partial \delta v_j^{*, \rho, l, k-1}}{\partial x_m} \right|_{L^2 \times H^{m-1}} \times \\
& \times \left| \frac{\partial v_m^{*, \rho, l, k-1}}{\partial x_j} + \frac{\partial v_m^{*, \rho, l, k-2}}{\partial x_j} \right|_{L^2 \times H^{m-1}}
\end{aligned}$$

From (88) and (89) we get

$$\begin{aligned}
& \left| \frac{\partial \delta v_i^{*, \rho, l, k}}{\partial \tau} \right|_{L^2 \times H^{m-2}} \leq \rho_l \nu n \max_{j \in \{1, \dots, n\}} \left| \delta v_i^{*, \rho, l, k} \right|_{L^2 \times H^{m-2}} \\
& + \rho_l C_{5/2} n \max_{j \in \{1, \dots, n\}} \left| v_j^{*, \rho, l, k-1} \right|_{L^2 \times H^{m-2}} \left| \delta v_i^{*, \rho, l, k-1} \right|_{L^2 \times H^{m-2}} \\
& + \rho_l C_{5/2} n \max_{j \in \{1, \dots, n\}} \left| \delta v_j^{*, \rho, l, k-1} \right|_{L^2 \times H^{m-2}} \left| v_i^{*, \rho, l, k-1} \right|_{L^2 \times H^{m-2}} \quad (90) \\
& + \rho_l C_{5/2} C_K n^2 \max_{j, m \in \{1, \dots, n\}} \left| \frac{\partial \delta v_j^{*, \rho, l, k-1}}{\partial x_m} \right|_{L^2 \times H^{m-2}} \times \\
& \times \left( \left| v_m^{*, \rho, l, k-1} \right|_{L^2 \times H^{m-2}} + \left| v_m^{*, \rho, l, k-2} \right|_{L^2 \times H^{m-2}} \right)
\end{aligned}$$

Now from previous estimates we have the upper bound

$$\left| v_m^{*, \rho, l, k-1} \right|_{L^2 \times H^{m-2}} + \left| v_m^{*, \rho, l, k-2} \right|_{L^2 \times H^{m-2}} \leq 2C_{k-1} \quad (91)$$

for some constant  $C_{k-1} > 0$ , hence

$$\begin{aligned}
& \left| \frac{\partial \delta v_i^{*, \rho, l, k}}{\partial \tau} \right|_{L^2 \times H^{m-2}} \leq \rho_l \nu n \max_{j \in \{1, \dots, n\}} \left| \delta v_i^{*, \rho, l, k} \right|_{L^2 \times H^{m-2}} \\
& + \rho_l C_{5/2} (2n + n^2) \max_{j \in \{1, \dots, n\}} (1 + C_k) C_{k-1} \left| \delta v_i^{*, \rho, l, k-1} \right|_{L^2 \times H^{m-2}} \quad (92)
\end{aligned}$$

Furthermore, from our previous estimates and for some  $\rho_l^0$  we have contraction for the first term on the right side of (93). Hence for some  $\rho_l > 0$  independent of  $k$  we get

$$\left| \frac{\partial \delta v_i^{*, \rho, l, k}}{\partial \tau} \right|_{L^2 \times H^{m-2}} \leq \frac{1}{4} \max_{j \in \{1, \dots, n\}} (1 + C_k) C_{k-1} \delta v_i^{*, \rho, l, k-1} \Big|_{L^2 \times H^{m-2}} \quad (93)$$



Similar construction estimates can be obtained by analogous methods successively for higher mixed derivatives the functions

$$D_x^\alpha \frac{\partial \delta v_i^{*,\rho,l,k}}{\partial \tau} \quad (94)$$

(for multiindex  $\alpha$ ), then for

$$\frac{\partial^2 \delta v_i^{*,\rho,l,k}}{\partial \tau^2}, \quad (95)$$

and then successively for higher order mixed and higher order time derivatives.

## 5 Global linear bound of the Leray projection term, local and global solutions

First let us consider the inductive construction of local regular solutions on  $[l-1, l] \times \mathbb{R}^3$  by the scheme above. At each time step  $l \geq 1$  having constructed  $v_i^{*,\rho,l-1}(l-1, \cdot) \in C^m \cap H^m$  at time step  $l-1$  (at  $l=1$  these are just the initial data  $h_i$ ), as a consequence of the argument above we have a time-local pointwise limit  $v_i^{*,\rho,l}(\tau, \cdot) = v_i^{*,\rho,l-1}(\tau, \cdot) + \sum_{k=1}^{\infty} \delta v_i^{*,\rho,l,k}(\tau, \cdot) \in H^2$  for all  $1 \leq i \leq n$ , where for  $n=3$  we have  $H^2 \subset C^\alpha$  uniformly in  $\tau \in [l-1, l]$ . Furthermore the functions of this series are Hölder continuous with respect to time. Note that the first order time derivative  $\frac{\partial}{\partial \tau} v_i^{*,\rho,l,k}(\tau, \cdot)$  of these members exist in  $H^m$  as well for natural  $m > \frac{3}{2}$  as we proved in the previous section. We observed that  $v_i^{*,\rho,l,k} \in H^m \times H^{2m}$  can be obtained inductively for each  $m$  and this leads to full local regularity of the limit function of the local scheme. If we plug in the approximating function  $v_i^{*,\rho,l,k}(\tau, \cdot)$  into the local incompressible Navier-Stokes equation in the Leray projection form, then we have

$$\begin{aligned} & \frac{\partial v_i^{*,\rho,l,k}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{*,\rho,l,k}}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{*,\rho,l,k} \frac{\partial v_i^{*,\rho,l,k}}{\partial x_j} \\ & - \rho_l \sum_{j,m=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{j,m=1}^n \left( \frac{\partial v_m^{*,\rho,l,k}}{\partial x_j} \frac{\partial v_j^{*,\rho,l,k}}{\partial x_m} \right) (\tau, y) dy, \\ & = + \rho_l \sum_{j=1}^n \delta v_j^{*,\rho,l,k} \frac{\partial v_i^{*,\rho,l,k}}{\partial x_j} \\ & - \rho_l \sum_{j,m=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{j,m=1}^n \left( \frac{\partial v_m^{*,\rho,l,k}}{\partial x_j} \frac{\partial v_j^{*,\rho,l,k}}{\partial x_m} \right) (\tau, y) dy \\ & + \rho_l \sum_{j,m=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{j,m=1}^n \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} \frac{\partial v_j^{*,\rho,l,k-1}}{\partial x_m} \right) (\tau, y) dy \end{aligned} \quad (96)$$

As  $k \uparrow \infty$  the right side of (96) becomes

$$\begin{aligned} & \rho_l \sum_{j=1}^n \delta v_j^{*,\rho,l,k} \frac{\partial v_j^{*,\rho,l,k}}{\partial x_j} + \rho_l \sum_{j=1}^n v_j^{*,\rho,l,k} \frac{\partial \delta v_j^{*,\rho,l,k}}{\partial x_j} \\ & \rho_l \int_{l-1}^\tau \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(z-y) \right) \times \\ & \times \left( \sum_{m,j=1}^n \frac{\partial \delta v_j^{*,\rho,l,k-1}}{\partial x_m} \left( \frac{\partial v_m^{*,\rho,l,k-1}}{\partial x_j} + \frac{\partial v_m^{*,\rho,l,k-2}}{\partial x_j} \right) \right) (s,y) G_l(\tau-s, x-z) dy dz ds, \end{aligned} \quad (97)$$

and since  $\lim_{k \uparrow \infty} \delta v_j^{*,\rho,l,k}(\tau, x) = 0$  and  $\lim_{k \uparrow \infty} \frac{\partial \delta v_j^{*,\rho,l,k}}{\partial x_j} = 0$  for all  $(\tau, x) \in [l-1, l] \times \mathbb{R}^n$  pointwise by our local contraction result, and since  $v_j^{*,\rho,l,k}$  and  $\frac{\partial \delta v_j^{*,\rho,l,k}}{\partial x_j}$  are uniformly bounded, we observe that the right side expressed in (97) goes to zero pointwise such that

$$\lim_{k \uparrow \infty} v_i^{*,\rho,l,k} \in C^{1,2}([l-1, l] \times \mathbb{R}^n), \quad 1 \leq i \leq n \quad (98)$$

satisfies the local Navier-Stokes equation pointwise, and in a classical sense. Finally in order to show that the Leray projection term is globally linearly bounded, it suffices to show that the *squared* function

$$l \rightarrow |v_j^{*,\rho,l}(l, \cdot)|_{H^2}^2 \quad (99)$$

grows linearly with the time step number  $l$ . Now we can set up a controlled scheme similar as in [1] and in [2], i.e., to consider a scheme

$$v_j^{*,\rho,l,k} = v_j^{*,\rho,l,k} + r_j^l \quad (100)$$

for some functions  $r_j^l$  which have a uniform upper bound with respect to some regular norm  $|\cdot|_{H^m \times H^{2m}}$ . Especially this upper bound should be independent of the time step number  $l$ . We shall see that it is possible to simplify the control function considered in [1] and [2]. Observe that the contraction results allow us to estimate the higher correction terms

$$v_j^{*,\rho,l} - v_j^{*,\rho,l,0} = \sum_{k=1}^{\infty} \delta v_j^{*,\rho,l,k} \quad (101)$$

in the local functional series on  $[l-1, l] \times \mathbb{R}^n$  at time step  $l \geq 1$  in terms of the function

$$v_j^{*,\rho,l,0} - v_j^{*,\rho,l-1}(l-1, \cdot), \quad (102)$$

i.e., the growth behavior of the latter function in (102) determines the growth behavior of the former function (101) with respect to the relevant norm by the related contraction result. Assume for a moment that we can prove that

$$\left| v_j^{*,\rho,l,0} - v_j^{*,\rho,l-1}(l-1, \cdot) \right| \lesssim \frac{1}{\sqrt{l}}, \quad (103)$$

and assume inductively that

$$\max_{j \in \{1, \dots, n\}} |v_j^{*, \rho, l-1}(l-1, \cdot)|_{H^2}^2 \leq C + (l-1)C, \quad (104)$$

which holds for  $l = 1$  and for  $\max_{j \in \{1, \dots, n\}} |h_j|_{H^2}^2$  for an appropriate constant for sure. Then assuming  $C \geq 4$  w.l.o.g. we have

$$\max_{j \in \{1, \dots, n\}} |v_j^{*, \rho, l-1}(l-1, \cdot)|_{H^2} \leq \sqrt{C + (l-1)C} \leq \frac{1}{2}(C + (l-1)C). \quad (105)$$

Now we have for some  $m \geq 2$

$$|\delta v_i^{*, \rho, l, k}|_{H^m \times H^{2m}}^2 \leq \frac{1}{4} \max_{j \in \{1, \dots, n\}} |\delta v_j^{*, \rho, l, k-1}|_{H^m \times H^{2m}}^2, \quad (106)$$

where we may choose  $\rho_l \sim \frac{1}{l}$  while for an appropriate choice of the constant factor (which transforms  $\sim$  to  $=$ ) we have

$$C_{k-1} \leq (\sqrt{C + C(l-1)} + 1) \text{ for all } k. \quad (107)$$

Assuming that we can realize the bound

$$\left| v_j^{*, \rho, l, 0} - v_j^{*, \rho, l-1}(l-1, \cdot) \right| \leq \frac{1}{2C\sqrt{l}}, \quad (108)$$

we can choose  $\rho_l$  such that we have indeed a contraction estimate with contraction constant  $\frac{1}{2C\sqrt{l}}$ . We get

$$\begin{aligned} & \max_{j \in \{1, \dots, n\}} |v_j^{*, \rho, l}|_{H^m \times H^{2m}} \\ & \leq \max_{j \in \{1, \dots, n\}} |v_j^{*, \rho, l-1}(l-1, \cdot) + \sum_{p=0}^{\infty} \delta v_j^{*, \rho, l, p}|_{H^m \times H^{2m}} \\ & \leq \max_{j \in \{1, \dots, n\}} |v_j^{*, \rho, l-1}(l-1, \cdot)|_{H^m \times H^{2m}} + \sum_{p=0}^{\infty} |\delta v_j^{*, \rho, l-1, k}|_{H^m \times H^{2m}} \\ & \leq \sqrt{C + (l-1)C} + \frac{1}{C\sqrt{l}}. \end{aligned} \quad (109)$$

Hence,

$$\begin{aligned} & \max_{j \in \{1, \dots, n\}} |v_j^{*, \rho, l}(l, \cdot)|_{H^m \times H^{2m}}^2 \\ & \leq (\sqrt{C + (l-1)C} + \frac{1}{C\sqrt{l}})^2 = C + (l-1)C + \sqrt{C + (l-1)C} \frac{1}{C\sqrt{l}} + \frac{1}{C^2 l} \\ & \leq C + lC. \end{aligned} \quad (110)$$

Hence, if we can realize the estimate

$$\left| v_j^{*, \rho, l, 0} - v_j^{*, \rho, l-1}(l-1, \cdot) \right| \leq \frac{1}{2C\sqrt{l}}, \quad (111)$$

then the scheme is global. This leads to the idea to extend the scheme for  $v_j^{*,\rho,l,k}$  to a controlled scheme for

$$v_j^{r,*,\rho,l,k} = v_j^{*,\rho,l,k} + r_j^l, \quad (112)$$

where

$$r_j^l = - \left( v_j^{*,\rho,l,0} - v_j^{*,\rho,l-1}(l-1, \cdot) \right). \quad (113)$$

In this case we have to ensure that we can repeat the contraction estimates above with this simplified control function. Note that each time step  $l \geq 1$  we first compute  $v^{*,\rho,l,0}$  as before via

$$\begin{cases} \frac{\partial v_i^{*,\rho,l,0}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{*,\rho,l,0}}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{*,\rho,l-1}(l-1, \cdot) \frac{\partial v_i^{*,\rho,l-1}}{\partial x_j}(l-1, \cdot) = \\ \rho_l \sum_{j,m=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{j,m=1}^n \left( \frac{\partial v_m^{*,\rho,l-1}}{\partial x_j} \frac{\partial v_j^{*,\rho,l-1}}{\partial x_m} \right) (l-1, y) dy, \\ \mathbf{v}^{*,\rho,l,0}(l-1, \cdot) = \mathbf{v}^{*,\rho,l-1}(l-1, \cdot). \end{cases} \quad (114)$$

We then define with  $r_j^l$  as in (113)

$$v_i^{r,*,\rho,l,0} = v_i^{*,\rho,l,0} + r_i^l. \quad (115)$$

If we plug in this control function into the local Navier-Stokes equation a time-step  $l \geq 1$  we get for each  $1 \leq i \leq n$  two additional terms  $F(\mathbf{v}^{r,*,\rho,l}, r_i^l)$  and  $G(r_i^l)$  where  $F$  is a bilinear functional of  $\mathbf{v}^{r,*,\rho,l}$  and  $r_i^l$ , and  $G$  depends only on the control function. At each time step  $l \geq 1$  we have to solve for

$$\begin{cases} \frac{\partial v_i^{r,*,\rho,l}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{r,*,\rho,l}}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{r,*,\rho,l-1}(l-1, \cdot) \frac{\partial v_i^{r,*,\rho,l-1}}{\partial x_j}(l-1, \cdot) = \\ \rho_l \sum_{j,m=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{j,m=1}^n \left( \frac{\partial v_m^{r,*,\rho,l-1}}{\partial x_j} \frac{\partial v_j^{r,*,\rho,l-1}}{\partial x_m} \right) (l-1, y) dy \\ + F(\mathbf{v}^{r,*,\rho,l}, r_i^l) + G(r_i^l), \\ \mathbf{v}^{r,*,\rho,l}(l-1, \cdot) = \mathbf{v}^{r,*,\rho,l-1}(l-1, \cdot). \end{cases} \quad (116)$$

Then the higher approximations  $v_i^{r,*,\rho,l,k}$  are defined via the Cauchy problems

$$\left\{ \begin{array}{l} \frac{\partial v_i^{r,*,\rho,l,k}}{\partial \tau} - \rho l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{r,*,\rho,l,k}}{\partial x_j^2} + \rho l \sum_{j=1}^n v_j^{r,*,\rho,l,k-1}(l-1, \cdot) \frac{\partial v_i^{r,*,\rho,l,k-1}}{\partial x_j}(l-1, \cdot) = \\ \rho l \sum_{j,m=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{j,m=1}^n \left( \frac{\partial v_m^{r,*,\rho,l,k-1}}{\partial x_j} \frac{\partial v_j^{r,*,\rho,l,k-1}}{\partial x_m} \right) (l-1, y) dy \\ + F(\mathbf{v}^{r,*,\rho,l,k-1}, r_i^l) + G(r^l), \\ \mathbf{v}^{r,*,\rho,l,k}(l-1, \cdot) = \mathbf{v}^{r,*,\rho,l-1}(l-1, \cdot). \end{array} \right. \quad (117)$$

Hence, for the higher order corrections  $\delta v_i^{r,*,\rho,l,k} = v_i^{r,*,\rho,l,k} - v_i^{r,*,\rho,l,k-1}$  we get the equations

$$\left\{ \begin{array}{l} \frac{\partial \delta v_i^{r,*,\rho,l,k}}{\partial \tau} - \rho l \nu \sum_{j=1}^n \frac{\partial^2 \delta v_i^{r,*,\rho,l,k}}{\partial x_j^2} + \rho l \sum_{j=1}^n v_j^{r,*,\rho,l,k-1}(l-1, \cdot) \times \\ \times \frac{\partial \delta v_i^{r,*,\rho,l,k-1}}{\partial x_j}(l-1, \cdot) + \rho l \sum_{j=1}^n \delta v_j^{r,*,\rho,l,k-1}(l-1, \cdot) \frac{\partial v_i^{r,*,\rho,l,k-1}}{\partial x_j}(l-1, \cdot) \\ = \rho l \sum_{j,m=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(x-y) \right) \times \\ \times \left( \sum_{m,j=1}^n \frac{\partial \delta v_j^{r,*,\rho,l,k-1}}{\partial x_m} \left( \frac{\partial v_m^{r,*,\rho,l,k-1}}{\partial x_j} + \frac{\partial v_m^{r,*,\rho,l,k-2}}{\partial x_j} \right) \right) (l-1, y) dy \\ + F(\delta \mathbf{v}^{r,*,\rho,l,k-1}, r_i^l), \\ \delta \mathbf{v}^{r,*,\rho,l,k}(l-1, \cdot) = 0. \end{array} \right. \quad (118)$$

Note that the  $G$  term cancels for the latter equations of functional increments, and we have only an additional linear operator  $F$  (in a more general context we have defined this  $F$  explicitly in [1]). Now for this scheme the contraction estimates above can be repeated with rather trivial modifications for these linear terms, and we can realize the estimate (111).

This leads to an alternative proof to the scheme with a control function as discussed [1] and [2], and it is at the same time a simplification of the scheme in [3], [4]<sup>1</sup>, which are all based on the Leray projection formulation in [5]. The scheme in [1] and [2] with a control function have the advantage that the time step size of the scheme may be kept constant and the  $H^2$  norm has a global upper bound for all time nonetheless. Numerically it may have also better stability.

<sup>1</sup>This scheme has to be modified similarly

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